(1) Cartan-Weyl-Hopf theorem: Let G be a compact Lie group. Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal <u>Abelian sub-algebra</u> [Not an ideal]

(We state & prove the Lie algebra version)

Then (a) Cartan $\forall X \in \mathfrak{g}$. $\exists a \in G$ $Ad(a)(X) \in \mathfrak{t}$

$a \in G$ $Ad(a)(X)$ → orbit of X under Ad-action of G

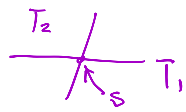
(b) If t_1, t_2 are two maximal Abelian subalgebras then $\exists g \in G$ $Ad(g)(t_1) = t_2$. ← Maybe Weyl

(c) Hopf: Let S be a torus sub group in G. Let g be an element ($\in G$) such that $g \in Z(S)$ - Centralizer of S.

Then $\exists T$ - a maximal torus. $T \supset S \cup \{g\}$.

In particular, $Z(S)$ is connected. In fact $Z(S) = \bigcup T$

T maximal torus $S \subset T$



Two key Lemmas.

<u>Kronecker Lemma</u>: For any torus group T. $\exists X \in t$ → Lie(T)

$g = \exp(X)$ satisfies the closure of $\{g^m, m \in \mathbb{Z}\}$ coincides T.

Ise-Takeuchi: P. 53-54. Reasonable given $T = S^1$

<u>Hopf Lemma</u>: Given any compact connected G, $\forall k$, positive integer $\forall g \in G$, $x^k = g$ has a solution. Namely $x^k = g$ can be solved in the group.

Ise-Takeuchi p 54-55

<u>Pf</u> (a) We first pick $X \in t$ such that $\overline{\exp(sX)} = T = \exp(t)$ the maximal torus.

[$\exp: \mathfrak{g} \to G_1$ is onto if G is compact due to the existence of bi-invariant metric]

For $Y \in \mathfrak{g}$. We consider $F(g) \doteq \langle X, Ad(g) Y \rangle$. A function on G

F attain its maximum somewhere say g_0. Then

$$\frac{d}{ds}\Big|_{s=0} \langle X, Ad(\exp(sZ)) Ad(g_0) Y \rangle = 0$$

$$\Rightarrow \quad \langle X, [Z, Ad(g_0)(Y)] \rangle = 0 \qquad \forall Z$$

$$\Rightarrow \quad \langle [Ad(g_0)(Y), X], Z \rangle = 0 \quad \forall Z$$ by invariance of $\langle , \rangle$.

$$\Rightarrow \quad [Ad(g_0)(Y), X] = 0.$$

This implies $Ad(g_0)(Y) \in \mathfrak{t}$, by the following claim

Claim: $\quad Z \in \mathfrak{t} \iff [Z, X] = 0 \qquad (*)$

Pf: $Z \in \mathfrak{t}$ iff $Ad(g)(Z) = Z \quad \forall g \in T. \qquad (**)$

Since clearly $\underset{\parallel\; e^{s\,ad_w}(Z)}{Ad(g)(Z)} = Z$ if $Z \in T$, $g = \exp(sw)$ if $w \in \mathfrak{t}$.

On the other hand $Ad(g)(Z) = Z \implies ad_w(Z) = 0 \quad \forall w \in \mathfrak{t}$

$\Rightarrow$ Span$\{\mathfrak{t}, Z\}$ is also Abelian $\Rightarrow Z \in \mathfrak{t}$. This proves $(**)$.

If $Z \in \mathfrak{t}$, $\underset{\parallel\; e^{s\,ad_X}(Z)}{Ad(\exp(sX))(Z)} = Z$ by the above $(**)$. $T \ni$

$\Rightarrow [X\ Z] = 0$

And, if $[X, Z] = 0 \Rightarrow Ad(\exp(sX))(Z) = Z$

$\Rightarrow Ad(T)(Z) = Z \Rightarrow Z \in \mathfrak{t}$ by the above

Namely $Ad(g)(Z) = Z$, $\forall g \in T$ -- by $(**)$. Hence the claim

The claim proves ⓐ

(b) Now pick $X_i \in t_i$ such that $\overline{\exp(sX_i)} = T_i = \exp(t_i)$

By (a) $\exists\ g_i$ $\quad Ad(g_i)(X_i) \in T_{i+1}$
$\qquad\qquad\qquad\qquad \overset{\shortparallel}{Y_{i+1}}$

Namely $\quad Y_2 = Ad(g_1)(X_1) \quad \in T_2$

$\qquad\qquad Y_1 = Ad(g_2)(X_2) \quad \in T_1$ $\qquad\qquad \varphi(\exp sW) = \exp(s\, d\varphi(W))$

$\Rightarrow \quad \underbrace{Ad(g_1^{-1})}(Y_2) = X_1 \Rightarrow a(g_1^{-1})(\exp(sY_2)) = \exp(sX_1)$

$\Rightarrow \quad \overline{a(g_1^{-1})(\exp(sY_2))} = \overline{\exp(sX_1)} = T_1$

Namely $T_1 \subset g_1^{-1} T_2\, g_1$, $g_1 T_1 g_1^{-1} \subset T_2 \Rightarrow \left.\begin{array}{l} Ad(g_1)(t_1) \subset t_2 \\ \text{the maximality of } T_1/t_1 \end{array}\right\} \Rightarrow Ad(g_1)(t_1) = t_2$

(c) Smaller is the group, the bigger is the centralizer.

Let $A = \overline{\{S, g\}}$ ← the group generated

A is Abelian, A_0 its connected component, which is a torus. Consider A/A_0 which is a finite group since G is compact.

If $g \in A_0$ we are done!

In general $[g]^m = 0$ (in A/A_0) for m large enough.

Let g' be the one generates A_0 $\quad (\overline{\{(g')^k\}} = A_0,\ k \in \mathbb{Z})$

Consider the equation $\qquad z^m = (g^m)^{-1} g'$

Solve it in A_0, with a solution z.

$\Rightarrow$ Let $x = gz \Rightarrow x^n = g^n \cdot z^n = g'$, which

generates A_0

Let B be the group generated by $\overline{\exp(sX)}$ with $\exp X = x$

B is an Abelian group.

$A_0 \subset B$ since $x^n = g'$

$\Rightarrow g \in B$ since $g = x z^{-1}$ $\in A_0$

Since $S \subset A_0$, $g \in B$

$\Rightarrow$ $S \cup \{g\} \subset B \subset$ the maximum torus contains B is defined as T.

It satisfies (c). $\square$

(2) Weyl group – $W(G)$.

Def: (a) $\dim(T)$, T a maximal torus in G, is called the rank of G.

(b) Let $N(T)$ be the normalizer of T. for a maximal torus T.

$W := N(T)/T$ is called the weyl group.

If T_1, T_2 are two maximal tori $\Rightarrow \exists g$ $gT_1g^{-1} = T_2$

Then $g(N(T_1))g^{-1} = N(T_2)$ since $gn_1g^{-1}(gt_1g^{-1})(gn_1g^{-1})^{-1}$

$= g\, \underbrace{n_1 t_1 n_1^{-1}}\, g^{-1} = g t_1' g^{-1} \in T_2$

& the other direction is similar.

Hence W (up to an isomorphism) is independent of the choice of T.

Properties of W:

Prop 3.35 of Ziller: (a) W is finite;

(b) W acts on $\mathfrak{t}$ via $w \cdot X = Ad(n)(X)$ for $w = [n]$, $n \in N(T)$, (is well-defined);

(c) The action is effective;

(d) If $Ad(G)(X)$ meets $\mathfrak{t}$ for $X \in \mathfrak{t}$ $\Rightarrow$ $\underset{[n] \in W}{\exists n}$ $Ad(n)(X) = Ad(g)(X)$.

Namely the orbits of X under the adjoint action intersects $\mathfrak{t}$, at the orbit of $Ad(w)(X)$.

(e) $Ad(G)(X)$ meets $\mathfrak{t}$ orthogonally, $\forall X \in \mathfrak{g}$.

Pf: (a) Note $Lie(N(T)) = \{ n \mid [n, z] \in \mathfrak{t},\ \forall z \in \mathfrak{t} \}$

Pick X, as in (1), consider $\langle \underbrace{[n, X]}_{\in \mathfrak{t}}, z \rangle$ $\forall z \in \mathfrak{t}$

$= \langle n, \underbrace{[X, z]}_{=0} \rangle \Rightarrow [n, X] = 0$

This implies $n \in \mathfrak{t}$ by $(*)$ in (1)

Hence $Lie(N(T)) = \mathfrak{t} \Rightarrow (N(T))_0 = T$

$N(T)/T$ is discrete in a compact space $\Rightarrow$ must be finite.

For (b) $n_1 = n_2 \cdot a$ $a \in T$

$Ad(n_1) = Ad(n_2) Ad(a) \Rightarrow Ad(n_1)(X) = Ad(n_2)(X)$

Since $Ad(a)(X) = X$ $a = \exp(tZ)$ $Ad(\exp(tZ))(X) = e^{t\, ad_Z}(X)$
$= X$

For (c) If $Ad(n)(X) = X$ $\forall X \in \mathfrak{t}$, $\Rightarrow$ $nan^{-1} = a$ $\forall a \in T$
$\Rightarrow n \in Z(T) = T$.

$\Rightarrow [n] = [e]$.